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# On a problem in combinatorics(Problems in Combinatorics)

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On a problem in combinatorics.

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# 1. Introduction.

PROBLEM. There are  $n$  boys and  $n$  girls. They are sitting at table, the boys looking at the girls, and the girls the boys. Each member of each group chooses one of the other group. A boy and a girl that choose each other, we call them a couple. i) What is probability of that the number of couples is  $k$  ( $0 \leq k \leq n$ )? ii) How behave the probability distribution of the random variable  $k$ , for increasing  $n$ ?

This problem was proposed by Prof. Hisatsugu\*) in statistics. More generally, we define the (fixed points) problem of  $A_1 \times \cdots \times A_\lambda$  by the problem of counting the number

$$H_{A_1 \times \cdots \times A_\lambda}(k) = \#\{(f_1, \dots, f_\lambda) \in A_1 \times \cdots \times A_\lambda; \varphi(f_\lambda \circ \cdots \circ f_1) = k\},$$

where

$$A_1, \dots, A_\lambda \subset N^N = \{f; f \text{ is a map from } N \text{ into } N\},$$

$$\varphi(f) = \#\{x \in N; f(x) = x\} \quad (\text{for } f \in N^N)$$

and  $N = \{1, \dots, n\}$ . Let  $H_\lambda(n, k) = H_{(N^N)^\lambda}(k)$ . It is plain that

$$H_\lambda(n, n) = (n!)^{\lambda-1}, \quad \sum_{k=0}^n H_\lambda(n, k) = n^{\lambda n}.$$

Our probability asked at i) is  $H_2(n, k)/n^{2n}$ . We have explicit formula

$$(1.1) \quad H_\lambda(n, k) = \frac{n^{\lambda n}}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} \left\{ \frac{(n)_{j+k}}{n_{j+k}} \right\}^\lambda,$$

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where  $(x)_k = x(x-1)\dots(x-k+1)$ ,  $(x)_0 = x^0 = 1$ . (For some values of  $H_\lambda(n, k)$ , see the table in the appendix.) Especially in the case of  $\lambda = 1$ , we have

$$H_1(n, k) = \binom{n}{k} (n-1)^{n-k}.$$

We consider only the problem of  $(N^N)^\lambda$  herein, however, we would think that is not vacant to compare with the problem of  $(N!)^\lambda$ , where  $N!$  is symmetric group on  $N$ . We get following

TABLE I.

problem of	$(N^N)^\lambda$	$(N!)^\lambda$ *)	Poisson d. with pm. 1	def.
number	$H_\lambda(n, k)$	$(n!)^{\lambda-1} D(n, k)$		
probability	$\frac{H_\lambda(n, k)}{n^\lambda n} = \frac{1}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} \left\{ \frac{(n)_{j+k}}{n^{j+k}} \right\}^\lambda$	$\frac{D(n, k)}{n!} = \frac{1}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!}$	$\frac{1}{e \cdot k!}$	$p(k)$
j-th moment	$\sum_{m=0}^j S(j, m) \left\{ \frac{(n)_m}{n^m} \right\}^\lambda$	$\sum_{m=0}^n S(j, m) **)$	$\hat{B}_j$ †	$\{k^j\}_p$
characteristic function	$\sum_{m=0}^n \frac{(e^{it}-1)^m}{m!} \left\{ \frac{(n)_m}{n^m} \right\}^\lambda$	$\sum_{m=0}^n \frac{(e^{it}-1)^m}{m!}$	$e^{e^{it}-1}$	$\{e^{ikt}\}_p$

\*) The problem of  $(N!)^\lambda$  is reduced to the problem of  $N!$ , that is well known as 'problème des rencontres' (see [1]).

\*\*)  $S(j, m)$  is a Stirling number of the second kind, that is defined by  $x^j = \sum_{m=0}^j S(j, m) (x)_m = \sum_{m=0}^{\infty} S(j, m) (x)_m$ .

†  $\hat{B}_j$  is the j-th Bell number, that is defined by  $\hat{B}_j = \sum_{m=0}^j S(j, m)$ .

Our purpose is to get an asymptotic expression of  $H_\lambda(n, k)$  for increasing  $n$ . We firstly obtain an infinite sum form

$$(1.2) \quad H_\lambda(n, k) = \frac{n^{\lambda n}}{e \cdot k!} \sum_{m=0}^{\infty} c_m(\lambda, n) d_m(k),$$

where

$$c_m(\lambda, n) = \frac{(-1)^m}{m!} \sum_{j=0}^m (-1)^j \binom{m}{j} \left\{ \frac{(n)_j}{n^j} \right\}^\lambda,$$

$$d_m(k) = \sum_{t=0}^m (-1)^{m-t} \binom{m}{t} (k)_t.$$

When we set

$$(1.3) \quad c_m(\lambda, n) = \sum_{\ell=0}^{\lambda m} \alpha_\lambda(\ell, m) n^{-\ell},$$

we have

$$(1.4) \quad \alpha_\lambda(0, 0) = 1,$$

$$(1.5) \quad c_m(\lambda, n) = \sum_{\ell=\lfloor \frac{m+1}{2} \rfloor}^{\lambda(m-1)} \alpha_\lambda(\ell, m) n^{-\ell} \quad (m > 0)$$

(for some values of  $\alpha_\lambda(\ell, m)$ , see the appendix). Observing (1.2) and (1.5), we may conjecture that

$$(1.6) \quad \frac{H_\lambda(n, k)}{n^{\lambda n}} = \frac{1}{e \cdot k!} \sum_{\ell=0}^{L-1} n^{-\ell} \sum_{\frac{\ell}{\lambda} \leq m \leq 2\ell} \alpha_\lambda(\ell, m) d_m(k) + O_{\lambda, L}(n^{-L}).$$

We get following

THEOREM. For  $\lambda = 1, 2$ , (1.6) is correct.

COROLLARY. If  $\lambda = 1$  or  $2$ ,  $H_\lambda(n, k)/n^{\lambda n}$  converges to  $1/e \cdot k!$ , uniformly for  $k = 0, 1, 2, \dots$ , as  $n$  tends to infinity.

EXAMPLE. By the theorem, we have the asymptotic expansion

$$H_2(n, 0) \approx \frac{n^{2n}}{e} \left( 1 - \frac{1}{n} - \frac{2}{3n^2} - \frac{1}{2n^3} + \dots \right).$$

If  $n = 10$ , we have

$$H_2(10, 0) = 32845 \ 29863 \ 68548 \ 28800,$$

$$\frac{n^{2n}}{e} \left( 1 - \frac{1}{n} - \frac{2}{3n^2} - \frac{1}{2n^3} \right) \Big|_{n=10} = 0.32845 \ 50277 \times 10^{20}.$$

## 2. Proof of (1.1).

$$\begin{aligned}
\sum_{k=j}^n \binom{k}{j} H_{\lambda}(n, k) &= \sum_{k=j}^n \binom{k}{j} \sum_{\substack{(f_1, \dots, f_{\lambda}) \in (N^N)^{\lambda} \\ \varphi(f_{\lambda} \circ \dots \circ f_1) = k}} 1 \\
&= \sum_{k=j}^n \sum_{\substack{(f_1, \dots, f_{\lambda}) \in (N^N)^{\lambda} \\ \varphi(f_{\lambda} \circ \dots \circ f_1) = k}} \sum_{\substack{A_1 \subset \{x \in N; f_{\lambda} \circ \dots \circ f_1(x) = x\} \\ \# A_1 = j}} 1 \\
&= \sum_{\substack{A_1 \subset N \\ \# A_1 = j}} \sum_{\substack{(f_1, \dots, f_{\lambda}) \in (N^N)^{\lambda} \\ \{x \in N; f_{\lambda} \circ \dots \circ f_1(x) = x\} \supset A_1}} 1 \\
&= \sum_{\substack{A_1 \subset N \\ \# A_1 = j}} \dots \sum_{\substack{A_{\lambda} \subset N \\ \# A_{\lambda} = j}} \sum_{\substack{(f_1, \dots, f_{\lambda}) \in (N^N)^{\lambda} \\ f_i(A_i) = A_{i+1} \ (i=1, \dots, \lambda-1) \\ (f_{\lambda}|A_{\lambda}) \circ \dots \circ (f_1|A_1) = I_{A_1}}} 1 \\
&= \left\{ \binom{n}{j} \right\}^{\lambda} (j!)^{\lambda-1} (n^{n-j})^{\lambda} \\
(1.2) \quad &= \frac{n^{\lambda n}}{j!} \left\{ \frac{(n)_j}{n^j} \right\}^{\lambda},
\end{aligned}$$

where  $f_i|A_i$  is a restriction of  $f_i$  to  $A_i$  and  $I_{A_1}$  is the identity map on  $A_1$ . Thus we have

$$\begin{pmatrix} \binom{0}{0} & \dots & \binom{n}{0} \\ & \ddots & \vdots \\ 0 & & \binom{n}{n} \end{pmatrix} \begin{pmatrix} H_{\lambda}(n, 0) \\ \vdots \\ H_{\lambda}(n, n) \end{pmatrix} = \begin{pmatrix} \frac{n^{\lambda n}}{0!} \left\{ \frac{(n)_0}{n^0} \right\}^{\lambda} \\ \vdots \\ \frac{n^{\lambda n}}{n!} \left\{ \frac{(n)_n}{n^n} \right\}^{\lambda} \end{pmatrix}.$$

Since

$$\begin{pmatrix} \binom{0}{0} & \cdots & \binom{n}{0} \\ & \ddots & \vdots \\ 0 & & \binom{n}{n} \end{pmatrix}^{-1} = \begin{pmatrix} \binom{0}{0} & -\binom{1}{0} & \binom{2}{1} & \cdots & (-1)^{n-0} \binom{n}{0} \\ & \binom{1}{1} & -\binom{2}{1} & & \vdots \\ & & \binom{2}{2} & & \vdots \\ & 0 & & & \vdots \\ & & & & (-1)^{n-n} \binom{n}{n} \end{pmatrix},$$

we have

$$\begin{aligned} H_\lambda(n, k) &= \sum_{j=k}^n (-1)^{j-k} \binom{j}{k} \frac{n^{\lambda n}}{j!} \left\{ \frac{(n)_j}{n^j} \right\}^\lambda \\ &= \frac{n^{\lambda n}}{k!} \sum_{j=k}^n \frac{(-1)^{j-k}}{(j-k)!} \left\{ \frac{(n)_j}{n^j} \right\}^\lambda = \frac{n^{\lambda n}}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} \left\{ \frac{(n)_{j+k}}{n^{j+k}} \right\}^\lambda. \end{aligned}$$

q.e.d.

### 3. Moments.

By (2.1), we have

$$\sum_{k=j}^n (k)_j \frac{H_\lambda(n, k)}{n^{\lambda n}} = \left\{ \frac{(n)_j}{n^j} \right\}^\lambda \quad (0 \leq j \leq n).$$

Since  $(k)_j = 0$  ( $k < j$ ), the right hand side may be replaced by  $\sum_{k=0}^n$  instead of  $\sum_{k=j}^n$ . Let it be so. Then the both sides vanish if  $j > n$ .

Hence we have, for all non-negative integer  $j$ ,

$$\sum_{k=0}^n (k)_j \frac{H_\lambda(n, k)}{n^{\lambda n}} = \left\{ \frac{(n)_j}{n^j} \right\}^\lambda$$

Thus, for  $j = 0, 1, 2, \dots$ , we have the  $j$ -th moment

$$\begin{aligned} (3.1) \quad \sum_{k=0}^n k^j \frac{H_\lambda(n, k)}{n^{\lambda n}} &= \sum_{k=0}^n \frac{H_\lambda(n, k)}{n^{\lambda n}} \sum_{m=0}^j S(j, m) (k)_m \\ &= \sum_{m=0}^j S(j, m) \sum_{k=0}^n (k)_m \frac{H_\lambda(n, k)}{n^{\lambda n}} = \sum_{m=0}^j S(j, m) \left\{ \frac{(n)_m}{n^m} \right\}^\lambda. \end{aligned}$$

## 4. Characteristic function.

We consider the function  $\varphi_{\lambda,n}(t)$  that is called characteristic function of the probability distribution

$$\Phi_{\lambda,n}(I) = \sum_{k \in I} \frac{H_{\lambda}(n,k)}{n^{\lambda n}} \quad (I \subset N).$$

We have, by (2.1),

$$\begin{aligned} (4.1) \quad \varphi_{\lambda,n}(t) &\stackrel{\text{def.}}{=} \sum_{k=0}^n e^{ikt} \frac{H_{\lambda}(n,k)}{n^{\lambda n}} = \sum_{k=0}^n \frac{H_{\lambda}(n,k)}{n^{\lambda n}} \sum_{m=0}^k \binom{k}{m} (e^{it}-1)^m \\ &= \sum_{m=0}^n (e^{it}-1)^m \sum_{k=m}^n \binom{k}{m} \frac{H_{\lambda}(n,k)}{n^{\lambda n}} = \sum_{m=0}^n \frac{(e^{it}-1)^m}{m!} \left\{ \frac{(n)_m}{n^m} \right\}^{\lambda}. \end{aligned}$$

(3.1) could be deduced also from (4.1), by using the formula

$$\frac{(e^x-1)^m}{m!} = \sum_{j=m}^{\infty} S(j,m) \frac{x^j}{j!} \quad (\text{see [1]}).$$

Since  $\varphi_{\lambda,n}(t)/\exp(e^{it}-1)$  is an entire function of  $e^{it}-1$ , we may put

$$(4.2) \quad e^{-z} \varphi_{\lambda,n}(t) = \sum_{n=0}^{\infty} c_m(\lambda, n) z^m \quad (z = e^{it}-1).$$

Then, by (4.1), we have

$$\begin{aligned} (4.3) \quad c_m(\lambda, n) &= \sum_{j=0}^{\min(m,n)} \frac{1}{j!} \left\{ \frac{(n)_j}{n^j} \right\}^{\lambda} (-1)^{m-j} \frac{1}{(m-j)!} \\ &= \frac{(-1)^m}{m!} \sum_{j=0}^m (-1)^j \binom{m}{j} \left\{ \frac{(n)_j}{n^j} \right\}^{\lambda}, \end{aligned}$$

$$(4.4) \quad |c_m(\lambda, n)| \leq \frac{2^m}{m!}.$$

## 5. Proof of (1.2).

In this section, we shall show (1.2) with error term, that is the following

LEMMA 1.

$$\frac{H_\lambda(n, k)}{n^{\lambda n}} = \frac{1}{e k!} \sum_{m=0}^{M-1} c_m(\lambda, n) d_m(k) + E_M \quad (|E_M| \leq \frac{e^4 4^M}{M!}),$$

where

$$(4.5) \quad d_m(k) = \sum_{t=0}^m (-1)^{m-t} \binom{m}{t} (k)_t.$$

COROLLARY. (1.2) is valid.

PROOF. By Cauchy's theorem with (4.1), (4.2),

$$\begin{aligned} \frac{H_\lambda(n, k)}{n^{\lambda n}} &= \operatorname{Res}_{z=-1} \frac{\mathcal{P}_{\lambda, n}(t)}{(z+1)^{k+1}} = \sum_{m=0}^{\infty} c_m(\lambda, n) \operatorname{Res}_{z=-1} \frac{e^z z^m}{(z+1)^{k+1}} \\ &= \frac{1}{e} \sum_{m=0}^{M-1} c_m(\lambda, n) \operatorname{Res}_{z=0} \frac{e^z (z-1)^m}{z^{k+1}} + \frac{1}{2\pi i} \int_{|z|=p} \frac{e^{z-1}}{z^{k+1}} \sum_{m=M}^{\infty} c_m(\lambda, n) (z-1)^m dz. \end{aligned}$$

We have

$$\operatorname{Res}_{z=0} \frac{e^z (z-1)^m}{z^{k+1}} = \frac{d_m(k)}{k!},$$

where  $d_m(k)$  is defined by (4.5). On the other hand, by Taylor's theorem with (4.4),

$$\left| \sum_{m=M}^{\infty} c_m(n) x^m \right| \leq \sum_{m=M}^{\infty} \frac{2^m}{m!} |x|^m \leq \frac{|2x|^M}{M!} e^{2|x|}.$$

Therefore,

$$\begin{aligned} |E_M| &\leq \frac{1}{2\pi e} \int_{|z|=p} \frac{e^{\operatorname{Re} z}}{\rho^{k+1}} e^{2|z-1|} \frac{(2|z-1|)^M}{M!} |dz| \\ &\leq \frac{2^M}{M!} \rho^{-k} e^{3\rho+1} (\rho+1)^M. \end{aligned}$$

If we take  $\rho = 1$ , we have  $|E_M| \leq e^4 \cdot 4^M / M!$ . q.e.d.



6. On  $s_\lambda(j, \ell)$ .

Let

$$(6.1) \quad \{(x)_j\}^\lambda = \sum_{\ell=0}^{\lambda j} \Delta_\lambda(j, \ell) x^\ell = \sum_{\ell=-\infty}^{\infty} \Delta_\lambda(j, \ell) x^\ell$$

Especially,  $s(j, \ell) = s_1(j, \ell)$  is a Stirling number of the first kind. By the definition (6.1) of  $s_\lambda(j, \ell)$ , we have recurrence formulae

$$(6.2) \quad \begin{aligned} \Delta_\lambda(j+1, \ell) &= \sum_{\ell'=0}^{\lambda} (-1)^{\lambda-\ell'} \binom{\lambda}{\ell'} j^{\lambda-\ell'} \Delta_\lambda(j, \ell-\ell') \quad (j \geq 0), \\ \Delta_\lambda(j+1, \lambda(j+1)-\ell) &= \sum_{\ell'=0}^{\lambda} (-1)^{\ell'} \binom{\lambda}{\ell'} j^{\ell'} \Delta_\lambda(j, \lambda j - (\ell-\ell')) \quad (j \geq 0). \end{aligned}$$

The latter implies that  $s_\lambda(j, \lambda j - \ell)$  is a polynomial of degree  $2\ell$  in  $\mathbb{Q}[j]$ , for  $\ell \geq 0$ . So we may define  $\alpha_\lambda(\ell, m)$  by

$$(6.3) \quad \Delta_\lambda(j, \lambda j - \ell) = \sum_{m=0}^{2\ell} \alpha_\lambda(\ell, m) (j)_m$$

We define  $\alpha_\lambda(\ell, m) = 0$  outside  $\ell \geq 0, 0 \leq m \leq 2\ell$ . (6.2) leads us to

$$(6.4) \quad (m+1)\alpha_\lambda(\ell, m+1) = \sum_{\ell'=1}^{\lambda} (-1)^{\ell'} \binom{\lambda}{\ell'} \sum_{u=0}^{\ell'} \alpha_\lambda(\ell-\ell', m-u) \eta_{\ell', u}(m-u),$$

where

$$\eta_{\ell', u}(m) = \sum_{t=u}^{\ell'} \binom{\ell'}{t} m^{\ell'-t} S(t, u) \in \mathbb{Z}[m].$$

We now show the following

LEMMA 2. i)  $\alpha_\lambda(0, 0) = 1$ . ii) If  $\ell > 0$ , then  $\alpha_\lambda(\ell, 0) = 0$ . iii) For  $m > 0$ , if  $\ell > \lambda(m-1)$  then  $\alpha_\lambda(\ell, m) = 0$ .

PROOF. i) It is clear from (6.1) and (6.3). ii) If  $\ell > 0$ , by the definition (6.1) of  $s_\lambda(j, \ell)$ , we have  $\alpha_\lambda(0, -\ell) = 0$ . When we substitute  $j = 0$  in the both hand sides of (6.3), we obtain  $\alpha_\lambda(\ell, 0) = 0$ . iii) Induction on  $m$ . If  $m = 1$  and  $\ell > 0$ , we have, by (6.4),

$$\alpha_\lambda(\ell, 1) = \sum_{g=1}^{\lambda} (-1)^g \binom{\lambda}{g} \alpha_\lambda(\ell-g, 0) \eta_{g,0}(0).$$

Since  $\eta_{q,0}(0) = S(q, 0) = 0$  ( $q > 0$ ), we have  $\alpha_\lambda(\ell, 1) = 0$ . Suppose  $\alpha_\lambda(\ell, \mu) = 0$ , for  $\ell > \lambda(\mu - 1)$  and  $\mu = 1, \dots, m$ . Let  $\ell > \lambda m$ . If  $1 \leq q \leq \lambda$  and  $0 \leq u \leq q$ , we have

$$\ell - q > \lambda m - q \geq \lambda(m - u) - q \geq \lambda(m - u) - \lambda$$

and  $\alpha_\lambda(\ell - q, m - u) = 0$ . By (6.4) we get  $\alpha_\lambda(\ell, m + 1) = 0$ . q.e.d.

COROLLARY. If  $\ell > 0$ , then

$$(6.5) \quad \Delta_\lambda(j, \lambda j - \ell) = \sum_{\frac{\ell}{\lambda} + 1 \leq m \leq 2\ell} \alpha_\lambda(\ell, m) (j)_m.$$

7. On  $c_m(\lambda, n)$ .

We have

$$\left\{ \frac{(n)_j}{n^j} \right\}^\lambda = \frac{1}{n^{\lambda j}} \sum_{j=0}^m \Delta_\lambda(j, \ell) n^\ell = \sum_{\ell=0}^{\lambda j} \Delta_\lambda(j, \lambda j - \ell) n^{-\ell}.$$

By (4.3),

$$\begin{aligned} c_m(\lambda, n) &= \frac{(-1)^m}{m!} \sum_{j=0}^m (-1)^j \binom{m}{j} \sum_{\ell=0}^{\lambda j} \Delta_\lambda(j, \lambda j - \ell) n^{-\ell} \\ &= \frac{(-1)^m}{m!} \sum_{\ell=0}^{\lambda m} n^{-\ell} \sum_{\frac{\ell}{\lambda} \leq j \leq m} (-1)^j \binom{m}{j} \Delta_\lambda(j, \lambda j - \ell). \end{aligned}$$

If  $\ell = 0$ , we have

$$\sum_{\frac{\ell}{\lambda} \leq j \leq m} (-1)^j \binom{m}{j} \Delta_\lambda(j, \lambda j - \ell) = \sum_{j=0}^m (-1)^j \binom{m}{j} = \delta_{m,0}.$$

If  $\ell > 0$ ,

$$\begin{aligned} \sum_{\frac{\ell}{\lambda} \leq j \leq m} (-1)^j \binom{m}{j} \Delta_\lambda(j, \lambda j - \ell) &= \sum_{\frac{\ell}{\lambda} \leq j \leq m} (-1)^j \binom{m}{j} \sum_{\frac{\ell}{\lambda} + 1 \leq t \leq 2\ell} \alpha_\lambda(\ell, t) (j)_t \\ &= \sum_{\frac{\ell}{\lambda} + 1 \leq t \leq 2\ell} \alpha_\lambda(\ell, t) \sum_{\frac{\ell}{\lambda} \leq j \leq m} (-1)^j \binom{m}{j} (j)_t, \end{aligned}$$

since  $(j)_t = 0$  (if  $m < t$ ,  $j \leq m$ ),

$$= \sum_{\substack{\frac{\ell}{\lambda}+1 \leq t \leq 2\ell \\ t \leq m}} \alpha_\lambda(\ell, t) \sum_{\substack{\frac{\ell}{\lambda} \leq j \leq m}} (-1)^j \binom{m}{j} (j)_t,$$

here the parameter  $t$  runs in the range  $t > \ell/\lambda$ , so that,

$$= \sum_{\substack{\frac{\ell}{\lambda}+1 \leq t \leq 2\ell \\ t \leq m}} \alpha_\lambda(\ell, t) t! \sum_{j=t}^m (-1)^j \binom{m}{j} \binom{j}{t}$$

$$= \sum_{\substack{\frac{\ell}{\lambda}+1 \leq t \leq 2\ell \\ t \leq m}} \alpha_\lambda(\ell, t) t! (-1)^t \delta_{m,t}$$

$$= (-1)^m m! \alpha_\lambda(\ell, m).$$

We have

$$c_m(\lambda, n) = \sum_{\ell=0}^{\lambda m} n^{-\ell} \alpha_\lambda(\ell, m).$$

Thus (1.3)-(1.5) are consistent with (6.1)-(6.5).

## 8. Inequalities.

We provide several inequalities for proving the theorem.

LEMMA 3.

$$(8.1) \quad |d_m(k)| \leq 2^m k!,$$

$$(8.2) \quad |c_m(1, n)| \leq n^{-\lfloor \frac{m+1}{2} \rfloor},$$

$$(8.3) \quad |c_m(2, n)| \leq n^{-\lfloor \frac{m+1}{2} \rfloor} (m+1)!.$$

PROOF. (8.1) is clear from (4.5). To prove (8.2) and (8.3), we introduce functions

$$f_{\lambda, m}(x) = \sum_{\ell=0}^{\lambda m} \alpha_\lambda(\ell, m) x^\ell$$

of  $x$ . Then we have  $c_m(\lambda, n) = f_{\lambda, m}(1/n)$ . Since

$$|\alpha_\lambda(\ell, m)| = (-1)^\ell \alpha_\lambda(\ell, m) \quad (\text{see (6.4)}),$$

if  $m > 0$ , we have

$$|c_m(\lambda, n)| \leq \sum_{\ell=\lfloor \frac{m+1}{2} \rfloor}^{\lambda(m-1)} |\alpha_\lambda(\ell, m)| n^{-\ell} \leq n^{-\lfloor \frac{m+1}{2} \rfloor} f_{\lambda, m}(-1).$$

Let  $\beta_{\lambda, m} = f_{\lambda, m}(-1)$ . By (6.4), we have

$$(m+1) f_{\lambda, m+1}(x) = \sum_{q=1}^{\lambda} (-x)^q \binom{\lambda}{q} \sum_{u=0}^q \eta_{q, u}(m-u) f_{\lambda, m-u}(x),$$

$$(8.4) \quad m \beta_{1, m} = \beta_{1, m-2} + (m-1) \beta_{1, m-1},$$

$$(8.5) \quad m \beta_{2, m} = \beta_{2, m-3} + (2m-1) \beta_{2, m-2} + (m^2-1) \beta_{2, m-1}.$$

It is enough to show  $\beta_{1, m} \leq 1$  and  $\beta_{2, m} \leq (m+1)!$ . It follows from (8.4) that

$$\beta_{1, m} = \sum_{j=0}^m \frac{(-1)^j}{m!} \leq 1.$$

We now prove  $\beta_{2, m} \leq (m+1)!$  by induction.  $\beta_{2, 0} = 1 \leq 1!$ ,  $\beta_{2, 1} = 0 \leq 2!$ ,  $\beta_{2, 2} = 3/2 \leq 3!$ . Suppose  $m \geq 3$  and  $\beta_{2, \mu} \leq (\mu+1)!$  for  $\mu < m$ . Thus, by

(8.5), we have

$$\begin{aligned} \beta_{2, m} &\leq \frac{1}{m} \left\{ (m-2)! + (2m-1)(m-1)! + (m-1)(m+1)! \right\} \\ &= (m+1)! \left\{ \frac{1}{(m+1)m^2(m-1)} + \frac{2-\frac{1}{m}}{(m+1)m} + \frac{m-1}{m} \right\} \\ &\leq (m+1)! \left\{ \frac{1}{4 \cdot 3 \cdot 2 m} + \frac{2}{m^2} + 1 - \frac{1}{m} \right\} \\ &= (m+1)! \left\{ 1 - \frac{1}{m} \left( \frac{23}{24} - \frac{2}{m} \right) \right\} \leq (m+1)!. \end{aligned}$$

q.e.d.

## 9. Proof of the theorem.

We divide the series (1.2) into three parts as

$$\frac{H_\lambda(n, k)}{n^{\lambda n}} = \frac{1}{e k!} \sum_{m=0}^{2\lambda L} c_m(\lambda, n) d_m(k) + \frac{1}{e k!} \sum_{m=2\lambda L+1}^{[x]} c_m(\lambda, n) d_m(k) + E_{[x]+1}.$$

By Lemma 1, we have

$$(9.1) \quad |E_{[x]+1}| \leq e^4 \frac{4^{x+1}}{\Gamma(x+1)}.$$

By (1.4), (1.5) and (8.1), we obtain

$$(9.2) \quad \frac{1}{e k!} \sum_{m=0}^{2\lambda L} c_m(\lambda, n) d_m(k) = \frac{1}{e k!} \sum_{\ell=0}^{L-1} n^{-\ell} \sum_{\substack{\ell \\ \lambda \leq m \leq 2\ell}} \alpha_\lambda(\ell, m) d_m(k) + O_{\lambda, L}(n^{-L}).$$

By Lemma 3, for  $\lambda = 1, 2$ , we have

$$(9.3) \quad \frac{1}{e k!} \left| \sum_{m=2\lambda L+1}^{[x]} c_m(\lambda, n) d_m(k) \right| \leq e^{-1} n^{-(\lambda L+1)} \sum_{m=0}^{[x]} 2^m A_\lambda(m),$$

where  $A_\lambda(m) = 1$  if  $\lambda = 1$ ,  $= (m+1)!$  if  $\lambda = 2$ .

Suppose  $\lambda = 1$ . Then

$$(9.4) \quad e^{-1} n^{-(L+1)} \sum_{m=0}^{[x]} 2^m \leq e^{-1} 2^{x+1} n^{-(L+1)}.$$

For a sufficiently large  $n$ , there exists  $x = x(n) > 0$  such that

$$(9.5) \quad n^{-\mu} = \frac{2^x}{\Gamma(x+1)},$$

where  $\mu = L+1$ . If we set  $x = x(n)$  as (9.5), by Stirling's formula,

$$n^\mu = 2^{-x} \Gamma(x+1) = 2^{-x} \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} (1+o(1)) \quad (n \rightarrow \infty),$$

$$\mu \log n = \frac{1}{2} \log 2\pi + (x+\frac{1}{2}) \log x - x(1+\log 2) + o(1),$$

$$\mu \log n \sim x \log x,$$

$$(9.6) \quad \log \mu + \log \log n = \log x + \log \log x + o(1).$$

Hence we get

$$(9.7) \quad \log \log n \sim \log x.$$

Let  $x = y(n) \log n$ . Then by (9.6),

$$\log y = \log x - \log \log n = \log \mu - \log \log x + o(1),$$

$$y \sim \frac{\mu}{\log x}.$$

By (9.7), we have

$$(9.8) \quad x \sim \mu \frac{\log n}{\log \log n}.$$

(9.3)-(9.5) and (9.8) lead us to

$$\begin{aligned} \frac{1}{e k!} \sum_{m=2L+1}^{[x]} c_m(1, n) d_m(k) &= O(n^{-\mu} 2^x) \\ &= O(n^{-\mu} 2^{\mu \frac{\log n}{\log \log n} (1+o(1))}) = O(n^{-\mu+o(1)}) = O_L(n^{-L}). \end{aligned}$$

Similarly by (9.1), (9.5), (9.8), we have

$$E_{[x]+1} = O(n^{-\mu} 2^x) = O_L(n^{-L}).$$

In the case of  $\lambda = 1$ , we get the theorem.

Suppose  $\lambda = 2$ . We have

$$\sum_{m=0}^{[x]} 2^m (m+1)! \leq 2^{[x]} [x+1]! \sum_{m=0}^{[x]} 2^{-m} \leq 2^{x+1} \Gamma(x+2).$$

Let set

$$n^{-\mu} = \frac{2^x}{(x+1)\Gamma^2(x+1)} \quad (\mu = 2L+1),$$

for large  $n$ . On a similar way, we shall get

$$x \sim \frac{\mu}{2} \frac{\log n}{\log \log n}.$$

Thus we obtain

$$\begin{aligned}
& \frac{1}{e^k k!} \sum_{m=4L+1}^{\lfloor x \rfloor} c_m(2, n) d_m(k) = O(n^{-\mu} 2^x \Gamma(x+2)) \\
& = O(n^{-\frac{\mu}{2}} 2^{\frac{3}{2}x} \sqrt{x}) = O_{\mu}(n^{-\mu/2 + o(1)} \sqrt{\frac{\log n}{\log \log n}}) = O_L(n^{-L}), \\
& E_{\lfloor x \rfloor + 1} = O\left(\frac{4^x}{\Gamma(x+1)}\right) = O(n^{-\mu/2} 2^{\frac{3}{2}x} \sqrt{x}) = O_L(n^{-L}).
\end{aligned}$$

q.e.d.

#### Reference

- [1] Riordan, An introduction to combinatorial analysis, Wiley, 1958.
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## APPENDIX

Table of  $H_\lambda(n,k)$ . $H_1(n,k)$ 

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	1	2	1						
3	8	12	6	1					
4	81	108	54	12	1				
5	1024	1280	640	160	20	1			
6	15625	18750	9375	2500	375	30	1		
7	279936	326592	163296	45360	7560	756	42	1	
8	5764801	6588344	3294172	941192	168070	19208	1372	56	1

 $H_2(n,k)$ 

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	0	1					
2	2	12	2				
3	156	423	144	6			
4	16920	33185	13968	1440	24		
5	2764880	4581225	2088800	316200	14400	120	
6	650696400	973830816	460350000	85521600	6231600	151200	720

 $H_3(n,k)$ 

$n \backslash k$	0	1	2	3	4	5
0	1					
1	0	1				
2	4	56	4			
3	2880	13959	2808	36		
4	3392064	10139392	3100032	145152	576	
5	7258985600	16544150125	6178856000	526644000	8928000	14400



Table of  $\alpha_\lambda(\ell, m)$ .
$$\alpha_1(\ell, m)$$

$\ell \backslash m$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	0	$-\frac{1}{2}$								
2	0	0	0	$\frac{1}{3}$	$\frac{1}{8}$						
3	0	0	0	0	$-\frac{1}{4}$	$-\frac{1}{6}$	$-\frac{1}{48}$				
4	0	0	0	0	0	$\frac{1}{5}$	$\frac{13}{72}$	$\frac{1}{24}$	$\frac{1}{384}$		
5	0	0	0	0	0	0	$-\frac{1}{6}$	$-\frac{11}{60}$	$-\frac{17}{288}$	$-\frac{1}{144}$	$-\frac{1}{3840}$

$$\alpha_2(\ell, m)$$

$\ell \backslash m$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	0	-1								
2	0	0	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{1}{2}$						
3	0	0	0	-2	-4	$-\frac{5}{3}$	$-\frac{1}{6}$				
4	0	0	0	$\frac{2}{3}$	$\frac{59}{8}$	$\frac{337}{30}$	$\frac{185}{36}$	$\frac{5}{6}$	$\frac{1}{24}$		
5	0	0	0	0	$-\frac{11}{2}$	$-\frac{83}{3}$	$-\frac{827}{24}$	$-\frac{241}{15}$	$-\frac{29}{9}$	$-\frac{5}{18}$	$-\frac{1}{120}$